> **Remark** (Why the gamma function?)**.** The gamma distribution is the *parent* of both the exponential distribution and the chi-squared distribution, and it is closely tied to the Poisson process. Most of the work is pure-maths revision – integration by parts and reduction arguments.

Extending the Factorial

The factorial $n!$ is defined for non-negative integers. Is there a natural function of a *real* variable that passes through the factorials? Euler found one, written as an integral:

Definition. The **gamma function** is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$$

> **Remark** (Why $x > 0$?)**.** Near $t = \infty$ the factor e^{-t} crushes any power of t, so the tail always converges. Near $t = 0$ the integrand behaves like t^{x-1}, and $\int_0^1 t^{x-1}\,\mathrm{d}t$ converges if and only if $x > 0$. (With complex analysis the definition can be extended to all complex numbers except $0, -1, -2, \ldots$ – see the closing remark.)

Theorem (The functional equation)

For all $x > 0$,

$$\Gamma(x+1) = x\,\Gamma(x)$$

Example

Prove this, using integration by parts.

Example

Show that $\Gamma(1) = 1$, and deduce that $\Gamma(n) = (n-1)!$ for every positive integer n.

Example (Class practice)

Evaluate $\displaystyle\int_0^\infty x^5 e^{-x}\,\mathrm{d}x$ and $\displaystyle\int_0^\infty x^3 e^{-2x}\,\mathrm{d}x$.

Half-Integer Values

What is $\Gamma\left(\frac{1}{2}\right)$ – morally, “$\left(-\frac{1}{2}\right)!$”? The answer is one of the most famous in mathematics.

Fact (The Gaussian integral) —

$$\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$$

(Sketch of the standard proof: call the integral I; then $I^2 = \iint e^{-(u^2+v^2)}\,\mathrm{d}u\,\mathrm{d}v$ over the whole plane, which in polar coordinates becomes $\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \times \frac{1}{2} = \pi$. This is also the fact that makes the normal pdf integrate to 1.)

Theorem

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example

Prove this, using the substitution $t = u^2$.

Example

Evaluate $\Gamma\left(\frac{7}{2}\right)$.

Example (Class practice)

Evaluate $\Gamma\left(\frac{9}{2}\right)$ and $\displaystyle\int_0^\infty \sqrt{x}\, e^{-x}\, \mathrm{d}x$.

The integrand $x^n e^{-x^2}$ below should look familiar: it is exactly what the substitution $t = u^2$ produced in the proof of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, only here over a finite interval.

Example (CAIE FP1, November 2002)

It is given that, for $n \geq 0$,

$$I_n = \int_0^1 x^n e^{-x^2}\, \mathrm{d}x.$$

(i) Find I_1 in terms of e.

(ii) Show that

$$I_{n+2} = \frac{n+1}{2} I_n - \frac{1}{2e}.$$

(iii) Find I_5 in terms of e.

The Gamma Distribution

The gamma function lets us build a whole family of pdfs. First, generalise the substitution trick from earlier:

Lemma

For $\alpha, \beta > 0$,

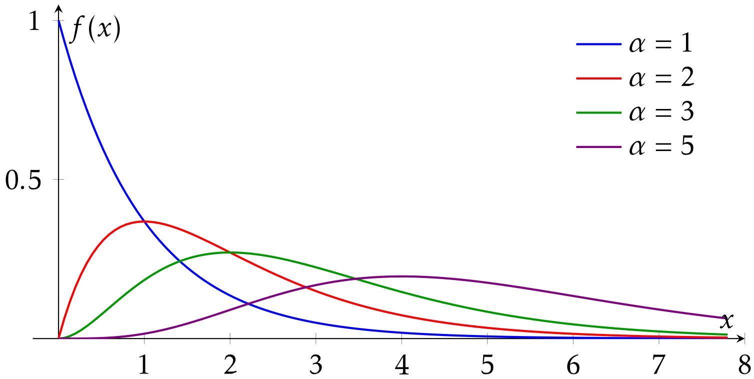
$$\int_0^\infty x^{\alpha-1} e^{-\beta x}\,\mathrm{d}x = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

The proof is a one-line substitution.

Dividing through by $\Gamma(\alpha)/\beta^\alpha$ gives a non-negative function with total integral 1 – that is, a pdf:

Definition. The random variable X has the **gamma distribution** with *shape* $\alpha > 0$ and *rate* $\beta > 0$, written $X \sim \Gamma(\alpha, \beta)$, if

$$f(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



Gamma pdfs with rate $\beta = 1$: for $\alpha = 1$ we get exponential decay; for larger α a skewed hump that drifts right and becomes more symmetric.

Theorem

If $X \sim \Gamma(\alpha, \beta)$ then

$$\mathbb{E}[X] = \frac{\alpha}{\beta} \qquad \text{and} \qquad \mathrm{Var}[X] = \frac{\alpha}{\beta^2}$$

Example

Prove this. (No integration by parts needed – use the lemma and the functional equation.)

Special Cases and the Poisson Process

Fact —

- **Exponential:** $\Gamma(1,\lambda)$ has pdf $\dfrac{\lambda^1}{\Gamma(1)}x^0e^{-\lambda x} = \lambda e^{-\lambda x}$, so $\mathrm{Exp}(\lambda) = \Gamma(1,\lambda)$. (Check: mean $\frac{1}{\lambda}$, variance $\frac{1}{\lambda^2}$. ✓)
- **Chi-squared:** the chi-squared distribution with k degrees of freedom (next chapter) is $\chi^2_k = \Gamma\left(\frac{k}{2},\frac{1}{2}\right)$ – which is why its pdf involves $\Gamma\left(\frac{k}{2}\right)$, and why half-integer values of Γ matter. Its mean is $\frac{k/2}{1/2} = k$ and variance $\frac{k/2}{1/4} = 2k$.

Theorem (Waiting time for the nth arrival)

In a Poisson process with rate λ, the waiting time T_n until the nth occurrence has distribution $\Gamma(n,\lambda)$.

Example

Prove this for general n, by finding $\mathbb{P}(T_n > t)$ and differentiating. (We did the case $n = 1$ – the exponential distribution – in the continuous random variables chapter.)

Example

A radioactive source emits particles as a Poisson process at a rate of 2 per second. Find the probability that the third particle is emitted within the first 2 seconds.

The MGF of the gamma distribution

If you studied the moment generating functions chapter, the gamma distribution ties everything together.

Example

Show that if $X \sim \Gamma(\alpha, \beta)$ then $M_X(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$ for $t < \beta$. Deduce that the sum of independent gamma variables with the *same rate* is gamma: $\Gamma(\alpha_1, \beta) + \Gamma(\alpha_2, \beta) \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

Remark (Shape–rate versus shape–scale)**.** We have used the *shape–rate* parametrisation $\Gamma(\alpha, \beta)$. Many texts (and most statistical software) instead use *shape–scale* parameters (α, θ) where $\theta = 1/\beta$, giving pdf $\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$, mean $\alpha\theta$ and variance $\alpha\theta^2$. Always check which convention a source is using before quoting formulae.

Remark (Further reading)**.** The gamma function is everywhere in higher mathematics. Extended to the complex plane it is intimately connected to the **Riemann zeta function** via

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,\mathrm{d}t$$

and it appears in the functional equation of ζ, at the heart of the Riemann Hypothesis – the most famous unsolved problem in mathematics. See [Toller] Ch 9 and any introduction to analytic number theory.

Textbook Exercises: [Toller] Ch 9